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ON THE SOLUTION OF A CLASS OF INTEGRAL EQUATIONS*

V.B. ZELENTSOV

A method is proposed for solving the integral equation of mixed problems of the theory of elasticity /1,2/. It is based on a special approximation of the kernel of integral equation. Solution of the integral equation is reduced to solving a finite algebraic system.

1. Integral equations arising in the course of solving plane static and dynamic mixed problems of the theory of elasticity in the case of semi-infinite bodies are of the form /1, 2/

$$\int_{-a}^{a} q(\xi)\,k(x-\xi)\,d\xi = 2\pi f(x), \quad |x| \leqslant a \tag{1.1}$$

$$k(t) = \int_{\gamma} K(\alpha)\,e^{-i\alpha t}\,d\alpha$$

where $q(x)$ are the unknown amplitudes of contact stresses, and $f(x)$ is a known function that defines the stamp shape. Function $K(u)$ has the properties defined in /1 - 4/ which include analyticity, parity, substance for real arguments, and the representation of that function in the form of a ratio of two entire functions and the retention at infinity the behavior of the form

$$K(u) \sim |u|^{-1} + O(|u|^{-3}), \quad |u| \to \infty$$

The contour γ is selected as in /5,6/.

For constructing an approximate solution of the obtained integral equation (1.1) function $K(u)$ is approximated by function $K_1(u)$ of the special form

$$K(\alpha) \approx K_1(\alpha) = \frac{\operatorname{th} A\alpha}{\alpha} \prod_{n=1}^{N} \frac{\alpha^2 + \delta_n'^2}{\alpha^2 + \gamma_n'^2} = \frac{\operatorname{th} A\alpha}{\alpha}\,\frac{P_1(\alpha^2)}{P_2(\alpha^2)} \tag{1.2}$$

where $A = \lim K(\xi)$ as $\xi \to 0$, $P_1(\alpha^2)$, $P_2(\alpha^2)$ are polynomials of power $2N$, and $\pm i\delta_n$ and $\pm i\gamma_n$ are zeros of these polynomials. The idea of such approximation for solving integral equation of the form (1.1) was first stated by Koiter and developed in /1,2/ and earlier works. It was shown in these that when $K(u)$ and $K_1(u)$ are close on the real axis, the solutions of Eq.(1.1) with kernels $K_1(u)$ and $K(u)$ are also close to each other.

Substituting approximation (1.2) into Eq.(1.1) and passing to dimensionless variables, we obtain

$$\int_{-1}^{1} \varphi(\xi)\,k^*(x-\xi)\,d\xi = 2\pi f(x), \quad |x| \leqslant 1; \quad \varphi(\xi) = aq(\xi) \tag{1.3}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\beta)\,e^{-i\beta x}\,d\beta = \begin{cases} \varphi(x), & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases}, \quad \Phi(\beta) = \int_{-1}^{1} \varphi(\xi)\,e^{i\beta\xi}\,d\xi$$

$$k^*(t) = \int_{\gamma} \frac{\operatorname{th}(A\beta\lambda)}{\beta} \prod_{n=1}^{N} \frac{\beta^2 + \delta_n^2}{\beta^2 + \gamma_n^2}\,e^{-i\beta t}\,d\beta$$

$$\delta_n = \frac{\delta_n'}{\lambda}; \quad \gamma_n = \frac{\gamma_n'}{\lambda}; \quad \lambda = \frac{h}{a}$$

Let us assume that the right-hand side of the integral equation (1.3) can be expanded in Fourier series. We solve this equation for the special right-hand side $e^{-\varepsilon x}$ (ε is a complex number), and represent $e^{-\varepsilon x}$ as the remainder of the even and odd functions

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$$f(x) = e^{-\varepsilon x} = \operatorname{ch} \varepsilon x - \operatorname{sh} \varepsilon x$$

2. For solving the integral equation (1.3) for the even right-hand side, it is sufficient to consider an equation of the form

$$\int_{-1}^{1} \varphi_+(\xi) k^*(x-\xi)\, d\xi = 2\pi \operatorname{ch} \varepsilon x, \quad |x| \leqslant 1 \tag{2.1}$$

Let $L = -d^2/dx^2$ and since $Le^{-i\beta x} = \beta^2 e^{-i\beta x}$, it is possible, using the expression in (1.2) for the kernel $K_1(\alpha)$, to transorm Eq.(2.1) to the form

$$P_1(L)\, \omega_+(x) = 2\pi P_2(L) \operatorname{ch} \varepsilon x = 2\pi P_2(-\varepsilon^2) \operatorname{ch} \varepsilon x, \quad |x| \leqslant 1 \tag{2.2}$$

$$\omega_+(x) = \int_{-1}^{1} \varphi_+(\xi)\, d\xi \int_{-\infty}^{\infty} \frac{\operatorname{th}(A\lambda\beta)}{\beta} \cos\beta(x-\xi)\, d\beta \tag{2.3}$$

where $P_1(L)$ and $P_2(L)$ are differential operators of order $2N$ in x.

Solution of the differential equation (2.2) for function $\omega_+(x)$ with allowance for its parity can be represented in the form /4/

$$\omega_+(x) = 2\pi \frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)} \operatorname{ch} \varepsilon x + 2\pi \sum_{n=1}^{N} C_n \operatorname{ch} \delta_n x \tag{2.4}$$

where C_n are unknown constants. Taking into account (2.3), we obtain for the determination of $\varphi_+(\xi)$ an integral equation which it is convenient to represent in the form of the paired integral equation

$$\int_{-\infty}^{\infty} \Phi_+(\beta) \frac{\operatorname{th}(A\alpha\beta)}{\beta} e^{-i\beta x}\, d\beta = 2\pi \frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)} \operatorname{ch} \varepsilon x + 2\pi \sum_{n=1}^{N} C_n \operatorname{ch} \delta_n x, \quad |x| \leqslant 1 \tag{2.5}$$

$$\int_{-\infty}^{\infty} \Phi_+(\beta) e^{-i\beta x}\, d\beta = 0, \quad |x| > 1$$

where $\Phi_+(\beta)$ is the same as in (1.3), where $\varphi_+(\beta)$ is to be substituted for $\varphi(\xi)$. Solution of this paired integral equation can be obtained on the basis of /7/. This solution contains unknown constants C_n that are obtained from the condition of fulfillment of the input integral equation (2.1).

We continue as follows. We obtain the Fourier transform $\Phi_+(\beta)$ of function $\varphi_+(x)$. Then taking into account the recurrent formulas for spherical functions and formula /8/

$$\int_a^b P_\nu(z) P_\sigma(z)\, dz = \frac{[\sqrt{z^2-1}\,(P_\sigma P_\nu^1 - P_\nu P_\sigma^1)]_a^b}{(\nu-\sigma)(\nu+\sigma+1)}$$

we obtain

$$\Phi_+(\beta) = \Omega_\varepsilon P_{-1/2+i\beta/\theta}(\operatorname{ch}\theta) - \frac{\pi^2\varepsilon^2 \operatorname{sh}\theta}{\beta^2+\varepsilon^2} \prod(i\beta, \varepsilon) - \pi \operatorname{sh}\theta \sum_{n=1}^{N} C_n \frac{\delta_n^2}{\beta^2+\delta_n^2} \prod(i\beta, \delta_n) \tag{2.6}$$

$$Q_{-1/2}\Omega_\varepsilon = \pi \frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)} R(\varepsilon, 0) \operatorname{sh}\theta - \operatorname{sh}\theta \sum_{n=1}^{N} C_n R(\delta_n, 0), \quad \theta = \frac{\pi}{A\lambda}$$

$$\prod(u, v) = P_{-1/2+u/\theta} P^1_{-1/2+v/\theta} - P_{-1/2+v/\theta} P^1_{-1/2+u/\theta}$$

$$R(u, v) = P_{-1/2+u/\theta} Q^1_{-1/2+v/\theta} - Q_{-1/2+v/\theta} P^1_{-1/2+u/\theta}$$

where $P_\nu^\mu = P_\nu^\mu(\operatorname{ch}\Theta)$, $Q_\nu^\mu = Q_\nu^\mu(\operatorname{ch}\Theta)$ are adjoint spherical functions of the first and second order, respectively. Integral representation of these functions /7 - 9/, differential relations between them, and also formulas linking elliptic integrals to spherical functions were used in the derivation of (2.6).

Taking into account (1.3) we substitute (2.6) into (2.1), and obtain a system of N algebraic equations with N unknown C_n. In calculating the resulting quadratures it is necessary to use the following integral representation of the associated Legendre functions of the first kind /10/:

$$P_{-1/2+i\tau}(\operatorname{ch}\alpha) = \frac{2}{\pi} \operatorname{th}\pi\tau \int_\alpha^\infty \frac{\sin\tau s}{\Delta(s, \alpha)}\, ds, \quad \Delta(u, v) = \sqrt{2(\operatorname{ch}u - \operatorname{ch}v)} \tag{2.7}$$

and of all functions used in the derivation of (2.6). After the determination of all quadratures we obtain the equation

$$\sum_{m=1}^{N} G_m \operatorname{ch} \gamma_m x = 0$$

which must be satisfied for any x from the interval $|x| \leqslant 1$. This condition yields for the determination of C_n the linear algebraic system

$$\sum_{n=1}^{N} x_n = f_m + \sum_{n=1}^{N} a_{mn} x_n, \quad m = 1, 2, \dots N \tag{2.8}$$

$$x_n = Q^1_{-1/2} P_{\nu n} C_n$$

$$a_{mn} = \frac{Q_{-1/2}}{Q^1_{-1/2}} \frac{H(\delta_n, \gamma_m)}{\delta_n^2 - \gamma_m^2}$$

$$f_m = \pi \varepsilon^2 \frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)} \frac{Q_{-1/2}}{Q_{\sigma m}} \frac{R(\varepsilon, \gamma_m)}{\varepsilon^2 - \gamma_m^2} - \pi \frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)} R(\varepsilon, 0)$$

$$H(u, v) = \frac{u^2 P_u Q_v^1 - v^2 P_u^1 Q_v}{P_u Q_v}$$

$$\nu_n = -\frac{1}{2} + \frac{\delta_n}{\theta}, \quad \sigma_m = -\frac{1}{2} + \frac{\gamma_m}{\theta}, \quad \theta = \frac{\pi}{A\lambda}$$

where $(P_\nu^\mu, Q_\nu^\mu R(u, v)$ are the same as in (2.6).

Having determined the constants C_n, we can write down the solution of the integral equation (2.1)

$$\varphi_+(x) = -\theta \frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)} \frac{\operatorname{sh}\theta}{Q_{-1/2}} \frac{R(\varepsilon, \theta)}{\Delta(\theta, \theta x)} - \frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)} \varepsilon^2 \int_x^1 P_{-1/2+\varepsilon\theta}(\operatorname{ch}\theta\tau) \operatorname{sh}\theta\tau \frac{d\tau}{\Delta(\theta\tau, \theta x)} - \tag{2.9}$$

$$\sum_{n=1}^{N} C_n \left[\frac{\theta \operatorname{ch}\theta}{Q_{-1/2}} \frac{R(\delta_n, 0)}{\Delta(\theta, \theta x)} + \delta_n^2 \int_x^1 P_{-1/2+\delta_n/\theta}(\operatorname{ch}\theta\tau) \operatorname{sh}(\theta\tau) \frac{d\tau}{\Delta(\theta\tau, \theta x)} \right]$$

where $(R(u, v), \Delta(u, v)$ are the same as in (2.6) and (2.7).

3. Let us solve the integral equation (1.3) for the odd function $f(x) = \operatorname{sh} \varepsilon x$ on the assumption that the right-hand side can be expanded in Fourier series. The integral equation assumes the form

$$\int_{-1}^{1} \varphi_-(\xi) k^*(x - \xi) d\xi = 2\pi \operatorname{sh} \varepsilon x, \quad |x| \leqslant 1 \tag{3.1}$$

where $k^*(t)$ is defined by formula (1.3). Using the differential operators $P_1(L)$ and $P_2(L)$, $L = -d^2/dz^2$ we reduce the integral equation (3.1) to the form

$$P_1(L)\,\omega_-(x) = 2\pi\, P_2(L) \operatorname{sh} \varepsilon x = 2\pi P_2(-\varepsilon^2) \operatorname{sh} \varepsilon x \tag{3.2}$$

where $\omega_-(x) = \omega_+(x)$ is defined in (2.3), and $P_1(L)$, $P_2(L)$ are the same as in (2.2). Solution of the differential equation (3.2) with allowance for its oddness can be represented in the form

$$\omega_-(x) = 2\pi \frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)} \operatorname{sh} \varepsilon x + 2\pi \sum_{n=1}^{N} D_n \operatorname{sh} \delta_n x$$

where D_n are unknown constants. Taking into account (2.3) we obtain for the determination of $\varphi_-(\xi)$ an integral equation whose solution can be derived using the data of /7/. In the considered here case we have

$$\varphi_-(x) = \varepsilon \frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)} \operatorname{sh}\theta x \left[\frac{P_{-1/2+\varepsilon/\theta}(\operatorname{ch}\theta)}{\Delta(\theta, \theta x)} - \theta \int_x^1 \frac{P_{-1/2+\varepsilon/\theta}(\operatorname{ch}\theta\tau)\, d\tau}{\Delta(\theta\tau, \theta x)} \right] + \tag{3.3}$$

$$\operatorname{sh}\theta x\sum_{n=1}^{N}D_n\delta_n\left[\frac{P_{-1/2+\delta_n/\theta}(\operatorname{ch}\theta)}{\Delta(\theta,\theta x)}-\theta\int_x^1\frac{P^1_{-1/2+\delta_n/\theta}(\operatorname{ch}\theta\tau)}{\Delta(\theta\tau,\theta x)}\,d\tau\right]$$

where $\Delta(u, v)$ is the same as in (2.7).

Constants D_n are determined by the condition that (3.3) must satisfy the integral equation (3.1). To substitute (3.3) into (3.1) it is necessary to know the Fourier transform of function $\varphi_-(x)$, i.e.

$$\Phi_-(\beta)=\int_{-1}^{1}\varphi(\xi)\,e^{i\beta\xi}\,d\xi$$

After the calculation of quadratures we obtain

$$\Phi_-(\beta)=\pi\varepsilon^2\frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)}\frac{\beta\operatorname{sh}\theta}{\beta^2+\varepsilon^2}\Pi(i\beta,\varepsilon)+\pi\beta\operatorname{sh}\theta\sum_{n=1}^{N}D_n\frac{\delta_n^2}{\delta_n^2+\beta^2}\Pi(i\beta,\delta_n) \tag{3.4}$$

where $\Pi(u, v)$ is the same as in (2.6).

Using (3.4) when substituting (3.3) into (3.1) we obtain for the determination of the unknown constants D_n the algebraic system

$$\sum_{n=1}^{N}a_{mn}x_n=f_m,\qquad n=1,2,\ldots,N \tag{3.5}$$

$$x_n=\delta_n^2D_n$$

$$a_{mn}=\frac{R(\delta_n,\gamma_m)}{\delta_n^2-\gamma_m^2}$$

$$f_m=2\varepsilon^2\frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)}\frac{R(\varepsilon,\gamma_m)}{\varepsilon^2-\gamma_m^2}$$

where $R(u, v)$ is the same as in (2.6).

Thus the solution of Eq.(3.1) is provided by formulas (3.3), and constants D_n are determined using (3.5).

4. Having determined the solutions of the integral equations (2.1) and (3.1) with special right-hand sides it is possible to construct the solution of the input integral equation (1.3) with the special right-hand side $e^{-\varepsilon x}$. The solution of that equation is defined by the formula

$$\varphi(x)=\varphi_+(x)-\varphi_-(x)$$

where $\varphi_+(x)$ and $\varphi_-(x)$ are calculated using formulas (2.9) and (3.3), respectively.

The contact problem integral characteristic (the force acting on the stamp) is determined by the formula

$$P=\int_{-1}^{1}\varphi_+(\xi)\,d\xi$$

which after the determination of quadrature assumes the form

$$2Q_{-1/2}P=\pi\frac{P_2(-\varepsilon^2)}{P_1(-\varepsilon^2)(\varepsilon-\theta)}[\varepsilon P_{-1/2+\varepsilon/\theta}+\theta\operatorname{sh}\theta P_{-1/2}R(\varepsilon,0)]+\pi\sum_{n=1}^{N}C_n\left[\delta_nP_{-1/2+\delta_n/\theta}+\frac{\theta\operatorname{sh}\theta\, P_{-1/2}R(\delta_n,\theta)}{\delta_n-\theta}\right] \tag{4.1}$$

In the important particular case of $\varepsilon=0$ (a flat stamp) and $f(x)=\text{const}$ the solution of integral equation (1.3) assumes the form

$$\varphi(x)=\frac{\theta}{Q_{-1/2}\Delta(\theta,\theta x)}-\sum_{n=1}^{N}C_n\left[\frac{\theta\operatorname{sh}\theta}{Q_{-1/2}}\frac{R(\delta_n,0)}{\Delta(\theta,\theta x)}+\delta_n^2\int_x^1\frac{P_{-1/2+\delta_n/\theta}\operatorname{ch}(\theta\tau)\operatorname{sh}(\theta\tau)}{\Delta(\theta\tau,\theta x)}\,d\tau\right] \tag{4.2}$$

Constants C_n are determined from an algebraic system of the type (2.8) in which

$$x_n=C_n\operatorname{sh}\theta Q^1_{-1/2}-P_{-1/2+\delta_n/\theta},\qquad f_m=\pi \tag{4.3}$$

and a_{mn} is the same as in (2.8).

In this case the integral characteristic is calculated using the derived from (4.1) formula

$$2Q_{-1/2}P = \pi P_{-1/2} + \pi \sum_{n=1}^{N} C_n \left[\delta_n P_{-1/2+\delta_n/\theta} + \frac{\theta \operatorname{sh} \theta P_{-1/2} R(\delta_n, \theta)}{\delta_n - \theta} \right]$$

In (4.1), (4.2), and (4.3) $R(u, v)$ is defined in (2.6) and $\Delta(u, v)$ in (2.7).

Note that examples of approximations of the form (1.2) for certain contact problems of the theory of elasticity can be found, for instance, in /1/.

5. As an example, we present the solution of the dynamic contact problem of bending a semi-infinite Kirchhoff plate of the form of a strip of height H.

The differential equation of plate bending is of the form

$$D\Delta\Delta W - \rho h \frac{\partial^2 W}{\partial t^2} = q(x, y, t) \tag{5.1}$$

where $W(x, y, t)$ is the plate deflection, D is its flexural rigidity, and $q(x, y, t)$ is the normal load. Let us assume $q(x, y, t) \equiv 0$. To investigate the plate harmonic oscillations we seek a solution of Eq.(5.1) of the form

$$W(x, y, t) = w(x, y) e^{-i\omega t} \tag{5.2}$$

Let the plate side faces be fixed as follows

$$\begin{aligned} & w(x, H) = M_y(x, 0) = V_y(x, 0) = 0, \quad |x| < \infty \\ & M_y(x, H) = 0, \ |x| > a \\ & w_y'(x, H) = \beta, \ |x| \leqslant a \\ & w(x, y) \to \infty \ \text{as} \ \sqrt{x^2 + y^2} \to \infty \end{aligned} \tag{5.3}$$

where M_y, V_y are, respectively, the bending moment and generalized shear force.

The substitution of (5.2) into (5.1) yeilds the equation

$$D\Delta\Delta w - \rho h \omega^2 w = 0 \tag{5.4}$$

Using the integral Fourier transformation we reduce the mixed boundary value problem (5.3), (5.4) to solving Eq.(1.3) (with $\beta = \text{const}$) and to Eq.(2.1) (when $\varepsilon = 0$) for the unknown reaction moment $\varphi(x)$ at the restraint. In that case function $K(u)$ is of the form

$$\begin{aligned} & K(u) = 2\varkappa^{-2} R(u) \{\sigma_1 \sigma_2 [(1-\nu)^2 u^4 - \nu^4] + \\ & \quad u^2 \operatorname{sh} \sigma_1 \operatorname{sh} \sigma_2 [(1-\nu)^2 u^4 - (1-2\nu)\varkappa^4] - \sigma_1 \sigma_2 \operatorname{ch} \sigma_1 \operatorname{ch} \sigma_2 [(1-\nu)^2 u^4 + \varkappa^4]\} \\ & R(u) = \{\sigma_1 [(1-\nu) u^2 - \varkappa^2]^2 \operatorname{ch} \sigma_1 \operatorname{sh} \sigma_2 - \sigma_2^2 [(1-\nu) u^2 + \varkappa^2]^2 \operatorname{sh} \sigma_1 \operatorname{ch} \sigma_2\}^{-1} \\ & \sigma_{1,2} = \sqrt{u^2 \pm \varkappa^2}, \quad \varkappa^2 = H^2 \omega \sqrt{\rho h D^{-1}} \end{aligned}$$

and has all properties defined in Sect.1.

With $\varkappa = 5$ the approximation of $K(u)$ of form (1.2) has the following parameters: $\delta_1 = 1.5863i$, $\delta_2 = 1.7454i$, $\delta_3 = 326.9100$, $\delta_4 = 15.5980$, $\delta_5 = 6.9051$, $\gamma_1 = 2.7352i$, $\gamma_2 = 4.8167i$, $\gamma_3 = 5.3570$, $\gamma_4 = 6.8634$, $\gamma_5 = 54729$, $A = 0.0009565$.

Such approximation enables us to calculate the reaction moment using the formulas (4.2), (4.3) and (4.4).

In concluding we point out that the adduced here solution of the integral equation (1.3) is valid for any $\lambda = h/a$.

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